$\boxed{1(a)}$

$$\underbrace{(2x-1)}_{M} + \underbrace{(3y+7)}_{N}\frac{dy}{dx} = 0$$

Let

$M(x,y) = 2x-1$
$N(x,y) = 3y+7$

these are continuous everywhere

We have

$$\frac{\partial M}{\partial x} = 2, \quad \frac{\partial M}{\partial y} = 0$$
$$\frac{\partial N}{\partial x} = 0, \quad \frac{\partial N}{\partial y} = 3$$

continuous everywhere

And $\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$.

Thus the equation is exact.

We need to find $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

That is we want to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2x-1 \quad (1) \\ \frac{\partial f}{\partial y} &= 3y+7 \quad (2) \end{aligned}}$$

Integrate ① with respect to x to get

$$f(x,y) = x^2 - x + g(y)$$

where $g(y)$ is constant with respect to x.

Now differentiate the above equation with respect to y to get:

$$\frac{\partial f}{\partial y} = g'(y)$$

Using ② this gives

$$3y + 7 = g'(y)$$

Integrating with respect to y gives

$$\frac{3y^2}{2} + 7y = g(y)$$

Thus,

$$f(x,y) = x^2 - x + g(y) = x^2 - x + \frac{3y^2}{2} + 7y$$

So a solution to the ODE is given implicitly by the equation

$$x^2 - x + \frac{3y^2}{2} + 7y = c$$

where c is a constant.

$\boxed{①\,(b)}$

$$\underbrace{5x+4y}_{M}+\underbrace{(4x-8y^3)}_{N}y'=0$$

Let

$$\left.\begin{aligned} M(x,y)&=5x+4y\\ N(x,y)&=4x-8y^3 \end{aligned}\right\}\text{ continuous everywhere}$$

Then,

$$\left.\begin{aligned} \frac{\partial M}{\partial x}&=5, \quad \frac{\partial M}{\partial y}=4\\ \frac{\partial N}{\partial x}&=4, \quad \frac{\partial N}{\partial y}=-24y^2 \end{aligned}\right\}\text{ continuous everywhere}$$

We have that

$$\frac{\partial M}{\partial y}=4=\frac{\partial N}{\partial x}$$

So, the ODE is exact.

We want to find $f(x,y)$ where

$$\frac{\partial f}{\partial x}=M(x,y)$$

$$\frac{\partial f}{\partial y}=N(x,y)$$

So we need to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 5x + 4y \quad (1) \\ \frac{\partial f}{\partial y} &= 4x - 8y^3 \quad (2) \end{aligned}}$$

Integrate (1) with respect to x to get:

$$f(x,y) = \frac{5x^2}{2} + 4yx + g(y)$$

where g is constant with respect to x.

Differentiate this equation with respect to y to get

$$\frac{\partial f}{\partial y} = 4x + g'(y)$$

Set this equal to (2) to get

$$4x + g'(y) = \frac{\partial f}{\partial y} = 4x - 8y^3$$

Thus,

$$g'(y) = -8y^3$$

So,

$$g(y) = -\frac{8y^4}{4} = -2y^4$$

Thus,

$$f(x,y) = \frac{5}{2}x^2 + 4yx + g(y)$$

$$= \frac{5}{2}x^2 + 4yx - 2y^4$$

So an implicit solution to the ODE is given by the equation

$$\frac{5}{2}x^2 + 4yx - 2y^4 = c$$

Where c is any constant.

$\boxed{1(c)}$

$$\underbrace{-(x+6y)}_{N}y' + \underbrace{(2x+y)}_{M} = 0$$

Let

$$\left.\begin{aligned} M(x,y) &= 2x+y \\ N(x,y) &= -x-6y \end{aligned}\right\} \text{continuous everywhere}$$

Then,

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2, \quad \frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} &= -1, \quad \frac{\partial N}{\partial y} = -6 \end{aligned}\right\} \text{continuous everywhere}$$

We have that

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= -1 \end{aligned}\right\} \text{not equal}$$

Thus, the equation is <u>not</u> exact.

1(d)

$$\underbrace{\frac{2x}{y}}_{M} - \underbrace{\frac{x^2}{y^2}}_{N} \cdot \frac{dy}{dx} = 0$$

Let

$$\left.\begin{aligned} M(x,y) &= \frac{2x}{y} = 2xy^{-1} \\ N(x,y) &= -\frac{x^2}{y^2} = -x^2y^{-2} \end{aligned}\right\} \text{continuous except when } y=0$$

Then

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y^{-1}, \quad \frac{\partial M}{\partial y} = -2xy^{-2} \\ \frac{\partial N}{\partial x} &= -2xy^{-2}, \quad \frac{\partial N}{\partial y} = 2x^2y^{-3} \end{aligned}\right\} \text{continuous except when } y=0$$

Note that

$$\left.\frac{\partial M}{\partial y} = -2xy^{-2} = \frac{\partial N}{\partial x}\right\} \text{equal when } y \neq 0$$

Thus the equation is exact.

The solution will end up existing where $y \neq 0$ because of the above continuity notes.

We want to find f where

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

So we need to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy^{-1} \quad (1) \\ \frac{\partial f}{\partial y} &= -x^2y^{-2} \quad (2) \end{aligned}}$$

Integrating (1) with respect to x gives

$$f(x,y) = x^2y^{-1} + h(y)$$

where $h(y)$ is constant with respect to x.

Differentiate with respect to y to get

$$\frac{\partial f}{\partial y} = -x^2y^{-2} + h'(y)$$

Set this equal to (2) to get

$$-x^2y^{-2} + h'(y) = \frac{\partial f}{\partial y} = -x^2y^{-2}$$

Thus,

$$h'(y) = 0$$

So, $h(y) = 0$. ←

you could put $h(y) = c$ where c is a constant but then this c would just merge with the constant in the solution below

Then,

$$f(x,y) = x^2y^{-1} + h(y) = x^2y^{-1}$$

So, a solution to the ODE is given by

$$\boxed{\frac{x^2}{y} = c}$$

where c is any constant.

1(e)

$$\underbrace{(2y^2x-3)}_{M}+\underbrace{(2yx^2+4)}_{N}y'=0$$

Let

$$\left.\begin{aligned} M(x,y) &= 2y^2x-3 \\ N(x,y) &= 2yx^2+4 \end{aligned}\right\}\text{ continuous everywhere}$$

Then

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y^2 \qquad \frac{\partial M}{\partial y} = 4yx \\ \frac{\partial N}{\partial x} &= 4yx \qquad \frac{\partial N}{\partial y} = 2x^2 \end{aligned}\right\}\text{ continuous everywhere}$$

And,

$$\frac{\partial M}{\partial y} = 4yx = \frac{\partial N}{\partial x}$$

So, the ODE is exact.

We must find f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2y^2x-3 \quad (1) \\ \frac{\partial f}{\partial y} &= 2yx^2+4 \quad (2) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N \end{aligned}}$$

Integrate ① with respect to x to get

$$f(x,y) = y^2x^2 - 3x + h(y)$$

where $h(y)$ is constant with respect to x.

Differentiate with respect to y to get

$$\frac{\partial f}{\partial y} = 2yx^2 + h'(y)$$

Set this equal to ② to get

$$2yx^2 + h'(y) = \frac{\partial f}{\partial y} = 2yx^2 + 4$$

Thus,

$$h'(y) = 4$$

So,

$$h(y) = 4y$$

Thus, $f(x,y) = y^2x^2 - 3x + h(y) = y^2x^2 - 3x + 4y$

So an implicit solution to the ODE is given by

$$y^2x^2 - 3x + 4y = c$$

where c is any constant.

$\boxed{1\,(f)}$ Consider

$$\underbrace{\left(2y-\frac{1}{x}+\cos(3x)\right)}_{N}\frac{dy}{dx}+\underbrace{\frac{y}{x^2}-4x^3+3y\sin(3x)}_{M}=0$$

Let

$$\left.\begin{aligned} M(x,y)&=yx^{-2}-4x^3+3y\sin(3x)\\ N(x,y)&=2y-x^{-1}+\cos(3x)\end{aligned}\right\}\text{ continuous everywhere except when } x=0$$

Then,

$$\left.\begin{aligned} \frac{\partial M}{\partial x}&=-2yx^{-3}-12x^2+9y\cos(3x)\\ \frac{\partial M}{\partial y}&=x^{-2}+3\sin(3x)\\ \frac{\partial N}{\partial x}&=x^{-2}-3\sin(3x)\end{aligned}\right\}\text{ continuous everywhere except when } x=0$$

$$\frac{\partial N}{\partial y}=2$$

Note that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ except at discrete points when $\sin(3x)=0$.

Thus the equation is <u>not</u> exact.

② (a) From problem ① above we saw that a solution to

$$(2x-1)+(3y+7)\frac{dy}{dx}=0$$

is given by the equation

$$x^2-x+\frac{3y^2}{2}+7y=c$$

We want the solution to satisfy $y(1)=2$.

plug $x=1, y=2$ into

$$x^2-x+\frac{3y^2}{2}+7y=c$$

to get

$$1^2-1+\frac{3(2)^2}{2}+7(2)=c$$

So,

$$20=c$$

Thus, a solution to the initial value problem is given by

$$x^2-x+\frac{3}{2}y^2+7y=20.$$

(2)(b) We are given that the equation

$$\underbrace{(e^x+y)}_{M} + \underbrace{(2+x+ye^y)}_{N}y' = 0$$

is exact.

Check:

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

equal

Let's find f where

$$\frac{\partial f}{\partial x} = e^x + y \quad (1)$$

$$\frac{\partial f}{\partial y} = 2 + x + ye^y \quad (2)$$

$$\frac{\partial f}{\partial x} = M$$

$$\frac{\partial f}{\partial y} = N$$

Integrate (1) with respect to x to get

$$f(x,y) = e^x + yx + h(y)$$

where h(y) is constant with respect to x.

Differentiate this equation with respect to y to get

$$\frac{\partial f}{\partial y} = x + h'(y)$$

Set this equal to ② to get

$$x + h'(y) = \frac{\partial f}{\partial y} = 2 + x + ye^y$$

Thus,

$$h'(y) = 2 + ye^y$$

So,

$$h(y) = 2y + \int ye^y \, dy$$

$$= 2y + ye^y - \int e^y \, dy$$

$u = y$ $du = dy$

$dv = e^y dy$ $v = e^y$

$\int u\,dv = uv - \int v\,du$

$$= 2y + ye^y - e^y$$

Thus,

$$f(x,y) = e^x + yx + h(y)$$

$$= e^x + yx + 2y + ye^y - e^y$$

So, an implicit solution to the ODE is given by the equation

$$e^x + yx + 2y + ye^y - e^y = c$$

where c is any constant.

We want the solution when $y(0) = 1$.

Plug in $x = 0,\ y = 1$ into the above to get

$$\underbrace{e^0}_{1} + \underbrace{1 \cdot 0}_{0} + \underbrace{2 \cdot 1}_{2} + \underbrace{1 \cdot e^1}_{e} - e^1 = c$$

Thus,

$$c = 3$$

So a solution to the initial value problem is given by

$$e^x + yx + 2y + ye^y - e^y = 3$$

② (c) We are given that the equation

$$\underbrace{\left(\frac{3y^2-x^2}{y^5}\right)}_{N}\frac{dy}{dx}+\underbrace{\frac{x}{2y^4}}_{M}=0$$

is exact.

Check:

$M=\frac{1}{2}xy^{-4} \rightarrow \frac{\partial M}{\partial y}=-2xy^{-5}$

$N=3y^{-3}-x^2y^{-5} \rightarrow \frac{\partial N}{\partial x}=-2xy^{-5}$

equal

We want f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2}xy^{-4} \quad (1) \\ \frac{\partial f}{\partial y} &= 3y^{-3}-x^2y^{-5} \quad (2) \end{aligned}}$$

$\frac{\partial f}{\partial x}=M$

$\frac{\partial f}{\partial y}=N$

Integrate ① with respect to x to get

$$f(x,y)=\frac{1}{4}x^2y^{-4}+h(y)$$

where h(y) is constant with respect to x.

Now differentiate the above with respect to y to get

$$\frac{\partial f}{\partial y} = -x^2 y^{-5} + h'(y)$$

Set this equal to 2 to get

$$-x^2 y^{-5} + h'(y) = \frac{\partial f}{\partial y} = 3y^{-3} - x^2 y^{-5}$$

Thus,

$$h'(y) = 3y^{-3}$$

So,

$$h(y) = \frac{3}{-2} y^{-2} = -\frac{3}{2} y^{-2}$$

Thus,

$$F(x,y) = \frac{1}{4} x^2 y^{-4} + h(y)$$

$$= \frac{1}{4} x^2 y^{-4} - \frac{3}{2} y^{-2}$$

So a solution to the ODE is given by

$$\frac{1}{4} x^2 y^{-4} - \frac{3}{2} y^{-2} = c$$

where c is any constant.

We want the solution when $y(1) = 1$.

So, plug $x=1, y=1$ into the above equation to get

$$\frac{1}{4}(1)^2(1)^{-4} - \frac{3}{2}(1)^{-2} = C$$

Thus,

$$C = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$$

So a solution to the initial value problem is given by

$$\frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} = -\frac{5}{4}$$

or

$$\frac{x^2}{4y^4} - \frac{3}{2y^2} = -\frac{5}{4}$$